Quantum Mechanics

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QUANTUM MECHANICS

Basics :- $E = h\nu$ - Planck's Equation.

where, E = energy radiated

h = planck's constant

ν = frequency of emitted radiation.

Values are, $h = 6.67 \times 10^{-34}$ Js.

→ $E = mc^2$ - Einstein's Equation.

where, E = energy.

m = mass of electron

c = velocity of light

value, $m = 9.1 \times 10^{-31}$ kg

$c = 3 \times 10^8$ m/s

According to Planck's Quantum Theory, the energy is radiated or absorbed in the form of discrete packets called 'quanta' or 'photon'. Each photon has energy equal to $h\nu$.

→ Frequency (ν) :- It is number of revolutions per second.

Combing both equations,

$$h\nu = mc^2$$

$$mc.c = h\nu$$

$$mc = \frac{h\nu}{c}$$

$\because$ $mc = p$ i.e. momentum but momentum of photon is independent of mass.

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$$\boxed{p = \frac{h\nu}{c}}$$ Expression for momentum of photon

∵ Photon has moving mass (and it is always moving and it is said to have zero rest mass).

Expression for mass is = $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ (Relativistic)

∵ velocity is more, $m >>> m_0$

$$p = \frac{h}{c/\nu} = \frac{h}{\lambda} \quad ∵ \lambda = \frac{c}{\nu}, \ \nu = \frac{c}{\lambda}$$

where, p = particle nature

λ = wave nature (wave length)

$$\therefore \boxed{p = \frac{h}{\lambda}}$$ De-Broglie Equation.

The above equation has both 'p' for particle property and 'λ' for wave property. So, it represents dual nature of matter.

$$\lambda = \frac{h}{p} = \frac{h}{mc} = \frac{h}{mv}$$

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Heiseinberg's Uncertainity Principle, (H.U.P):

According to H.U.P,

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2} \quad \left\{ \text{where, } \hbar = \frac{h}{2\pi} \right\}$$

here, p = momentum.

x = position of particle.

The physical meaning of expression is if you can determine the position of particle then there is uncertainity in determining momentum of particle and vice versa.

According to Classical Mechanics,

T.E = K.E + P.E

$$E_{total} = \frac{1}{2} Kx^2 + \frac{p^2}{2m}$$

As per H.U.P, 'p' and 'x' cannot be determin simultaneously implies K.E. & P.E. of a Quantum Mechanical Particle (Q.M.P.) cannot be determined simultaneously i.e. the Classical formula of T.E is no more valid for Q.M.P.

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Introduction to Quantum Mechanics.

Unlike in Classical Mechanics in Quantum we rely on the concept of "operator" any physical quantity like momentum, energy, K.E. are represented by there equivalent operators.

Few of them are :-

(1) Momentum Operator $\longrightarrow$ $-i\hbar \frac{\partial}{\partial x}$

(2) Energy Operator $\longrightarrow$ $-i\hbar \frac{\partial}{\partial t}$

(3) K.E. Operator $\longrightarrow$ $-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$

In Q.M, any system is represented by a wave function such as,

$$\psi = Ae^{-i(kx-\omega t)}$$

the above eq^n contains, all the possible information about the particle.

All the possible information mean its energy, momentum, wavelength in addition to where it located and at what time i.e. $(x \,\&\, t)$ OR (r, t).

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classmate

Example: Lets find the momentum of particle represented by wavefunction.

$$\psi = Ae^{-i(kx-\omega t)}$$

In order to find momentum we have to consider momentum i.e.,

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

$$-i\hbar \frac{\partial}{\partial x}\psi = -i\hbar \frac{\partial}{\partial x}\psi$$

$$-i\hbar \frac{\partial}{\partial x}\psi = -i\hbar \frac{\partial}{\partial x} Ae^{-i(kx-\omega t)}$$

$$= (-i\hbar)(-ik)\,Ae^{-i(kx-\omega t)}$$

$$= -\hbar k \psi$$

$$\Longrightarrow \boxed{\hat{p}\psi = -\hbar k \psi}$$

called as Eigen value equation of the operator $\hat{p}$ i.e. $\hbar k$ is the momentum of the particle. $\hbar k$ is the eigen value of operator '$\hat{p}$' which is actually precise measurement of that physical quantity whose operator is $\hat{p}$ i.e. momentum.

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Similarly,

$$-i\hbar \frac{\partial}{\partial t} \psi = \hbar \omega \psi$$

AND

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi = \frac{\hbar^2 k^2}{2m} \psi$$

gives energy and K.E. eigen value.

→ Schrodinger Equation.

Total energy in terms of operators can be expressed as,

$$K \cdot E = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}, \quad P \cdot E = V, \quad T \cdot E = -i\hbar \frac{\partial}{\partial t}$$

$$-i\hbar \frac{\partial}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \quad \ldots (1)$$

Operator alone has no meaning if not operated on a function. Let 'ψ' the function to be operated on, then the expression becomes,

$$-i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + V\psi \quad \ldots (2)$$

The expression above is known as, Schrodinger Time-Dependent Equation.

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If energy of the system is constant then,

$$E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V\psi$$

is defined as,

Schrodinger Time Independent Equation.

The above equation can be modified as,

$$\boxed{\frac{\partial^2}{\partial x^2}\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0}$$

In 3-D,

$$\boxed{\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0}$$

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ANGULAR MOMENTUM

Unlike in Classical Mechanics, in Quantum Mechanics in addition to Orbital Angular Momentum one also encounters Spin Angular Momentum.

Spin Angular Momentum is an intrinsic or elementary property of elementary particles such as electrons and photons. It has no Classical counterpart.

The operators correspond to Cartesian components of angular momentum in Quantum Mechanics obey a set of fixed fundamental Commutator Relations.

Eigen values of Angular momentum which emerges from this Commutator Relations are obtained.

Angular Momentum is related to position of particle and its linear momentum as below:

$$\vec{L} = \vec{r} \times \vec{p}$$

Let us consider $\vec{r} \times \vec{p}$,

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= \hat{i}\left[y p_z - z p_y\right] + \hat{j}\left[z p_x - x p_z\right] + \hat{k}\left[x p_y - y p_x\right]$$

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CLASSMATE

$$\Rightarrow L_x\hat{i} + L_y\hat{j} + L_z\hat{k} = \hat{i}[yP_z - zP_y] + \hat{j}[zP_x - xP_z] + \hat{k}[xP_y - yP_x].$$

i.e,

$$L_x = [yP_z - zP_y]$$

$$L_y = [zP_x - xP_z]$$

$$L_z = [xP_y - yP_x]$$

Above expression in operator form is expressible as,

$$\hat{L}_x = \hat{y}\hat{P}_z - \hat{z}\hat{P}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = \hat{z}\hat{P}_x - \hat{x}\hat{P}_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = \hat{x}\hat{P}_y - \hat{y}\hat{P}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$\hat{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z) = -i\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = -i\hbar\nabla$$

$$\therefore \boxed{\vec{L} = -i\hbar\,\vec{r} \times \vec{\nabla}}$$ Single Vector Eq^n

$\longrightarrow$ For understanding, (Commutator Relations)

$$(A-B)\,(A+B) = (A+B)\,(A-B)$$

But, $\left(\frac{\partial}{\partial x} - x\right)\left(\frac{\partial}{\partial x} + x\right) \neq \left(\frac{\partial}{\partial x} + x\right)\left(\frac{\partial}{\partial x} - x\right)$

$$\therefore \left(\frac{\partial}{\partial x} - x\right)\left(\frac{\partial}{\partial x} + x\right)\psi$$

$$\left(\frac{\partial}{\partial x} - x\right)\left(\frac{\partial}{\partial x}\psi + x\psi\right)$$

$$\frac{\partial^2}{\partial x^2}\psi + \cancel{x\frac{\partial \psi}{\partial x}} + \psi - x\frac{\partial}{\partial x}\psi - x^2\psi$$

$$\Longrightarrow \left(\frac{\partial^2}{\partial x^2} - x^2 + 1\right)\psi$$

$$\left(\frac{\partial}{\partial x} + x\right)\left(\frac{\partial}{\partial x} - x\right)\psi$$

$$\left(\frac{\partial}{\partial x} + x\right)\left(\frac{\partial \psi}{\partial x} - x\psi\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} - \cancel{x\frac{\partial \psi}{\partial x}} - \frac{\partial x}{\partial x} + x\frac{\partial \psi}{\partial x} - x^2\psi$$

$$\Longrightarrow \left(\frac{\partial^2}{\partial x^2} - x^2 - 1\right)\psi$$

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CLASSMATE

$$[\hat{x}\,\hat{p}_x]\psi = i\hbar\psi$$

$$[\hat{x}\hat{p}_x - \hat{p}_x\hat{x}]\psi$$

$$\left[\hat{x}\, i\hbar\frac{\partial}{\partial x} - i\hbar\frac{\partial}{\partial x}\hat{x}\right]\psi$$

$$\hat{x}\, i\hbar\frac{\partial}{\partial x}\psi - i\hbar\frac{\partial}{\partial x}(\hat{x}\psi)$$

$$\cancel{\hat{x}\, i\hbar\frac{\partial}{\partial x}\psi} - \cancel{i\hbar\hat{x}\frac{\partial}{\partial x}\psi} - \psi i\hbar\frac{\partial\hat{x}}{\partial x}$$

$$\therefore [\hat{x}\,\hat{p}_x]\psi = i\hbar\psi$$

Commutator Relations.

$$[\hat{L}_x, \hat{L}_y] = \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x$$

$$= (\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)(\hat{z}\hat{p}_y - \hat{y}\hat{p}_z) - (\hat{z}\hat{p}_x - \hat{x}\hat{p}_z)(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y)$$

$$= \hat{x}\hat{p}_y(\hat{z}\hat{p}_z - \hat{p}_z\hat{z}) - \hat{y}\hat{p}_x(\hat{z}\hat{p}_z - \hat{p}_z\hat{z})$$

$$= i\hbar(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)$$

$$= i\hbar\hat{L}_z$$

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Similarly,
$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$$
$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$$
$$[L_x, \hat{L}_y] = i\hbar \hat{L}_z$$

Single Vector Equation,
$$i\hbar \hat{L} = \hat{L} \times \hat{L}$$

In determinantal form,
$$i\hbar(e_x\hat{L}_x + e_y\hat{L}_y + e_z\hat{L}_z) = \begin{vmatrix} e_x & e_y & e_z \\ \hat{L}_x & \hat{L}_y & \hat{L}_z \\ \hat{L}_x & \hat{L}_y & \hat{L}_z \end{vmatrix}$$

The total angular momentum operator is the vector operator,
$$\hat{L} = e_x\hat{L}_x + e_y\hat{L}_y + e_z\hat{L}_z$$

from which we may form, $\hat{L}^2$.
$$\therefore \quad \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

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CLASSMATE

$$[\hat{L}_z, \hat{L}^2] = [\hat{L}_z, \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2]$$

$$= [L_z, L_x^2] + [L_z, L_y^2] + 0$$

$$= L_x [L_z, L_x] + [L_z, L_x] L_x + L_y [L_z, L_y] + [L_z, L_y] L_y$$

$$= -i\hbar [L_x L_y + L_y L_x - L_y L_x - L_x L_y]$$

$$= 0$$

Similarly, $\hat{L}_x$ and $\hat{L}_y$ also commute with $\hat{L}^2$.

In general,

$$[\hat{L}_x, \hat{L}^2] = [\hat{L}_y, \hat{L}^2] = [\hat{L}_z, \hat{L}^2] = 0$$

$$\therefore [\hat{L}, \hat{L}^2] = 0$$

Eigenfunction equations are as follows.

$$\hat{L}^2 = \hbar^2 l(l+1)$$

$$\hat{L}_z = \hbar m$$

If $n = 3$.

$l = 0, 1, 2$

$m = -2, -1, 0, 1, 2$

$$L = \sqrt{l(l+1)}\, \hbar$$

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The values that experiments finds are only of the form, $L = \hbar\sqrt{l(l+1)}$, where, l is some integer. For example, one would never measure the value, $L = \hbar\sqrt{7}$, since it is not of the form $L^2 = \hbar^2 l(l+1)$. There is no integer for which $l(l+1) = 7$. This is similar to fact that a particle in a 1-D box is never found to have the energy $E = 7E_1$.

This value does not fit the energy eigenvalue $E = n^2 E_1$.

If $l = 5$,

$$L^2 = 30\hbar^2$$

$$L^2 = \hbar^2 l(l+1)$$

It can be represented as,

$5\hbar, 4\hbar, 3\hbar, 2\hbar, 1\hbar, 0, -1\hbar, -2\hbar, -3\hbar, -4\hbar, -5\hbar$

$\therefore$ The eigenvalue,

$L^2 = \hbar^2 l(l+1)$ is $(2l+1)$ fold denerate i.e. 11-fold degenerate.

→ **Orbital Angular Momentum v/s Spin Angular Momentum.**

Orbital Angular momentum derives from the space & momentum co-ordinates of the particle & is similar to Classical $(\vec{r} \times \vec{p})$ angular momentum. In contrast, Spin angular momentum doesnot relat to a particle's co-ordinates or momenta nor are the eigen states of spin dependent on boundary conditions imposed in co-ordinate space.

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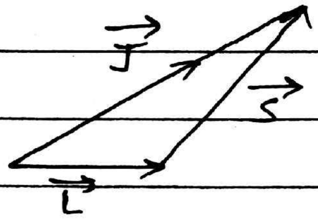
Spin as mentioned previously is an internal (intrinsic) property of an particle like mass or charge. It is an extra degree of freedom attached to a Quantum Mechanical Particle, and must be prescribed together with the values of all other compatible properties of a particle, inorder to designate state of particle. In particular spin momentum and hence spin wave functions donot have co-ordinate representations.

Eigen Values Of The Angular Momentum Operator:

$\hat{J}$ is used to denoted angular momentum in general while $\hat{L}$ will be reserved for Orbital Angular Momentum and $\hat{S}$ for Spin.

We know,

$$\vec{J} = \vec{L} + \vec{S}$$



$\vec{J}$ = total angular momentum

$\vec{L}$ = Orbital angular momentum

$\vec{S}$ = Spin momentum.

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z$$

$$[\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x$$

$$[\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y$$

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

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The component $\hat{J}$ obeys all rules what $\hat{L}$ obeys,

$$[\hat{J}_x, \hat{J}^2] = [J_y, \hat{J}^2] = [\hat{J}_z, \hat{J}^2] = 0$$

$$\Delta J_x \, \Delta J_y \geq \frac{\hbar}{2} |\langle J_z \rangle| \quad \{\text{from H.U.P}\}$$

Ladder Operators.

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \qquad (\text{Annhilation}) \; \ldots\ldots (a)$$

and

$$\hat{J}_- = \hat{J}_x - i\hat{J}_y \qquad (\text{Creation}) \; \ldots\ldots (b)$$

The above eq^n's (a) & (b) are called as Annhilation Operator and Creation Operator respectively.

$$\hat{J}_+ = \hat{J}_-^\dagger \qquad \{a^\dagger \text{ is complex conjuga[te] of } a.\}$$

$$\hat{J}_- = \hat{J}_+^\dagger$$

$$\left.\begin{aligned} [\hat{J}_z, \hat{J}_+] &= \hbar \hat{J}_+ \\ [\hat{J}_z, \hat{J}_-] &= -\hbar \hat{J}_- \end{aligned}\right\} [\hat{J}_z, \hat{J}_\pm] = \pm \hbar \hat{J}_\pm$$

$$\left.\begin{aligned} [\hat{J}^2, \hat{J}_+] &= 0 \\ [\hat{J}^2, \hat{J}_-] &= 0 \end{aligned}\right\} [\hat{J}^2, \hat{J}_\pm] = 0$$

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Lets prove : $[\hat{J}_z, \hat{J}_+] = \hbar \hat{J}_+$

Ans. Proof :-

$$[\hat{J}_z, \hat{J}_+] = [\hat{J}_z, \hat{J}_x + i\hat{J}_y]$$

$$= [\hat{J}_z, \hat{J}_x] + i[\hat{J}_z, \hat{J}_y]$$

$$= i\hbar \hat{J}_y - i \cdot i\hbar \hat{J}_x$$

$$= \hbar (\hat{J}_x + i\hat{J}_y)$$

$$= \hbar \hat{J}_+$$

Hence proved.

Similarly,

To prove :- $\hat{J}_z, \hat{J}_- = -\hbar \hat{J}_-$

Proof :- $[\hat{J}_z, \hat{J}_-] = [\hat{J}_z, \hat{J}_x - i\hat{J}_y]$

$$= [\hat{J}_z, \hat{J}_x] - i[\hat{J}_z, \hat{J}_y]$$

$$= i\hbar \hat{J}_y + i \cdot i\hbar \hat{J}_x$$

$$= i\hbar \hat{J}_y - \hbar \hat{J}_x$$

$$= -\hbar [\hat{J}_x - i\hat{J}_y]$$

$$= -\hbar \hat{J}_-$$

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- Few more relations through $\hat{J}_+$ & $\hat{J}_-$

$$\boxed{\hat{J}^2 = \hat{J}_\mp \hat{J}_\pm + \hat{J}_z^2 \pm \hbar \hat{J}_z}$$

i.e.

$$\hat{J}^2 = \hat{J}_- \hat{J}_+ + \hat{J}_z^2 + \hbar \hat{J}_z$$

$$= \hat{J}_+ \hat{J}_- + \hat{J}_z^2 - \hbar \hat{J}_z$$

Also, $[\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z$

- Few more relations through $\hat{J}_+$ & $\hat{J}_-$

$$\boxed{\hat{J}^2 = \hat{J}_\mp \hat{J}_\pm + \hat{J}_z^2 \pm \hbar \hat{J}_z} \quad \ldots \text{(A)}$$

i.e.,

$$\hat{J}^2 = \hat{J}_- \hat{J}_+ + \hat{J}_z^2 + \hbar \hat{J}_z \quad \text{— (1)}$$

$$= \hat{J}_+ \hat{J}_- + \hat{J}_z^2 - \hbar \hat{J}_z \quad \text{— (2)}$$

Also, $[\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z$.

By subtracting, eq^n (1) & (2),

$$\hat{J}^2 - \hat{J}_z^2 = \hat{J}_- \hat{J}_+ + \hbar \hat{J}_z$$

$$\hat{J}^2 - \hat{J}_z^2 = \hat{J}_+ \hat{J}_- - \hbar \hat{J}_z$$

$$0 = \hat{J}_- \hat{J}_+ - \hat{J}_+ \hat{J}_- = 2\hbar \hat{J}_z$$

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$$\therefore [\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z$$ //

By adding eq^n (1) & (2),

$$\hat{J}^2 - \hat{J}_z^2 = \hat{J}_- \hat{J}_+ + \hbar \hat{J}_z$$

$$\hat{J}^2 - \hat{J}_z^2 = \hat{J}_+ \hat{J}_- - \hbar \hat{J}_z$$

$$2(\hat{J}^2 - \hat{J}_z^2) = \hat{J}_+ \hat{J}_\pm + \hat{J}_\pm \hat{J}_+$$

i.e $2(\hat{J}^2 - \hat{J}_z^2) = \hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+$ //

Consider the relation,

$$\hat{J}^2 = (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y) + \hat{J}_z^2 + \hbar \hat{J}_z$$

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 + i(\hat{J}_x \hat{J}_y - \hat{J}_y \hat{J}_x) + \hbar \hat{J}_z$$

$$\hat{J}^2 = \hat{J}^2 + i(\hat{J}_x \hat{J}_y - \hat{J}_y \hat{J}_x) + \hbar \hat{J}_z$$

$$i(\hat{J}_x \hat{J}_y - \hat{J}_y \hat{J}_x) = -\hbar \hat{J}_z$$

Multiplying both sides by 'i', we get,

$$-(\hat{J}_x \hat{J}_y - \hat{J}_y \hat{J}_x) = -i\hbar \hat{J}_z$$

$$(\hat{J}_x \hat{J}_y - \hat{J}_y \hat{J}_x) = i\hbar \hat{J}_z$$

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z$$ //

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Let us consider the following expression,

$$\hat{J}_z \psi_m = \hbar m \psi_m \quad \ldots (1)$$

where, $\hbar m$ = eigen value of the operator $\hat{J}_z$ for a system represented by wave function ψ_m.

where, $\hbar m$ is the exact measurement of that physical quantity for which $\hat{J}_z$ is the operator of z component of total angular momentum.

$$\hat{J}_z \hat{J}_+ \psi_m = (\hbar \hat{J}_+ + \hat{J}_+ \hat{J}_z)\psi_m$$

$$= (\hbar \hat{J}_+ + \hat{J}_+ \hbar m)\psi_m \quad \ldots (2)$$

$$\hat{J}_z (\hat{J}_+ \psi_m) = \hbar(m+1)(\hat{J}_+ \psi_m)$$

i.e, $\hat{J}_+ \psi_m = \psi_{m+1} \quad \ldots (3)$

Applying $\hat{J}_+$ again gives,

$$\hat{J}_+(\hat{J}_+ \psi_m) = \hat{J}_+ \psi_{m+1} = \psi_{m+2} \quad \ldots (4)$$

Similarly,

$$\hat{J}_- \psi_m = \psi_{m-1}, \quad \hat{J}_- \psi_{m-1} = \psi_{m-2} \quad \ldots (5)$$

Thus, we have found the scheme of generating a sequence of unnormalized eigen functions of $\hat{J}_z$, from a single eigen function ψ_m. They are

$$(\ldots, \psi_{m-2}, \psi_{m-1}, \psi_m, \psi_{m+1}, \psi_{m+2}, \ldots) \quad \ldots (6)$$

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Since $\hat{J}^2$ commutes with $\hat{J}_z$, these operators have common eigen functions.

Let $\hbar^2 k^2$ be the eigen value of the operator $\hat{J}^2$, then

$$\hat{J}^2 \psi_m = \hbar^2 k^2 \psi_m \quad \ldots (7)$$

Operating the above eq^n (7) with $\hat{J}_+$, we get,

$$\hat{J}_+ \hat{J}^2 \psi_m = (\hbar^2 k^2) \hat{J}_+ \psi_m = \hat{J}^2 (\hat{J}_+ \psi_m) \ldots (8)$$

The last equality asserts that $\hat{J}_+ \psi_m = \psi_{m+1}$ is also an eigenfunction of $\hat{J}^2$.

i.e,

$$\hat{J}^2 \psi_{m+1} = \hbar^2 k^2 \psi_{m+1} \quad \ldots (9)$$

It follows that the sequence of eigen function $\hat{J}_z$ found previously are also eigenfunctions of $\hat{J}^2$ corresponding to same eigen value $\hbar^2 k^2$.

Also,

$$\langle J^2 \rangle = \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle = \hbar^2 k^2 \ldots (10)$$

But,

$$\langle J_z^2 \rangle = \hbar^2 m^2$$

So, above expression becomes,

$$\langle J^2 \rangle = \langle J_x^2 \rangle + \langle J_y^2 \rangle + \hbar^2 m^2 = \hbar^2 k^2 \quad \ldots (11)$$

This implies,

$$\hbar^2 k^2 \geq \hbar^2 m^2$$

$$\text{i.e,} \quad |k| \geq |m| \quad \ldots (12)$$

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For given value of $k > 0$, the possible values of m in the sequence eq^n 6, fall between $+k$ and $-k$.

If m_{max} is max. value that m can assume and m_{min} is min. value that m can assume then

$$\hat{J}_+ \psi_{max} = 0 \quad \ldots (13)$$

Similarly,

$$\hat{J}_- \psi_{min} = 0 \quad \ldots (14)$$

From eq^n (A), we get,

$$\hat{J}^2 \psi_{m\,max} = \hbar^2 k^2 \psi_{m\,max}$$

$$= \hat{J}_z^2 \psi_{m\,max} + \hbar \hat{J}_z \psi_{m\,max} \quad \ldots (15)$$

$$\hbar^2 k^2 = \hbar^2 m_{max}(m_{max} + 1) \quad \ldots (16)$$

Similarly, $\hat{J}^2 \psi_{m\,min} = \hbar^2 k^2 \psi_{m\,min}$

$$= \hat{J}_z^2 \psi_{m\,min} - \hbar \hat{J}_z \psi_{m\,min} \quad \ldots (17)$$

$$\hbar^2 k^2 = \hbar^2 m_{min}(m_{min} - 1) \quad \ldots (18)$$

i.e, $m_{max}(m_{max}+1) = m_{min}(m_{min}-1) \quad \ldots (19)$

which is satisfied if,

$$m_{max} = -m_{min} \quad \ldots (20)$$

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$m_{max} \equiv j \quad \ldots (21)$

Since m runs from $-j$ to $+j$, we get,

j = an integer if $m = 0$ is included in the sequence of m values

$j = \frac{1}{2} \times$ an odd integer if $m = 0$ is not included in sequence of m values.

$$J^2 = \hbar^2 k^2 = \hbar^2 j(j+1) \quad \ldots (22)$$

$$J^2 = \hbar^2 j(j+1)$$

$$J_z = \hbar m_j \quad (m_j = -j, \ldots, +j) \quad \ldots (23)$$

$$\hat{L}^2 \psi_{lm} = \hbar^2 l(l+1) \psi_{lm}$$

$$\left.\begin{aligned} \hat{L}_z \psi_{lm} &= \hbar m \psi_{lm} \quad (m = -l, \ldots +l) \\ \hat{L}_+ \psi_{lm} &= \psi_{l.m+1} \quad (\hat{L}_+ = \hat{L}_x + i\hat{L}_y) \\ \hat{L}_- \psi_{lm} &= \psi_{l.m-1} \quad (\hat{L}_- = \hat{L}_x - i\hat{L}_y) \end{aligned}\right\} \quad \ldots (24)$$

$$\left.\begin{aligned} L_+ \psi_{ll} &= 0 \\ L_- \psi_{l,-l} &= 0 \end{aligned}\right\} \quad \ldots (25)$$

$\therefore m = l$ to $m = -l$ i.e. min to max.

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- Eigen Functions of the Orbital Angular Momentum Operators $\hat{L}^2$ and $\hat{L}_z$.

Spherical Harmonics.

We know,

$$\hat{L}^2 \psi_{lm} = \hbar^2 l(l+1)\psi_{lm} \quad \ldots (1)$$

$$\hat{L}_z \psi_{lm} = \hbar m \psi_{lm} \quad \ldots (2)$$

$$\hat{L}_+ \psi_{ll} = 0 \quad \ldots (3)$$

The spherical coordinates (r, θ, ϕ) are related to the Cartesian coordinates (x, y, z) through the transformation equations,

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad \ldots (4)$$

The Cartesian components of $\hat{L}$ i.e,

$$\hat{L}_x = \hat{y}\hat{P}_z - \hat{z}\hat{P}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \ldots (5)$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \ldots (6)$$

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$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \quad \ldots (7)$$

We know,

$$\left.\begin{aligned} L_x &= yp_z - zp_y \\ L_y &= zp_x - xp_z \\ L_z &= xp_y - yp_x \end{aligned}\right\} \quad \ldots (8)$$

So,

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \quad \ldots (9)$$

$$\hat{L}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \quad \ldots (10)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \quad \ldots (11)$$

Using eq^n's (9), (10) & (11), we get first the ladder operators,

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y = \hbar e^{i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta}\right) \quad \ldots (12)$$

$$\hat{L}_- = \hat{L}_x - i\hat{L}_y = \hbar e^{-i\phi}\left(i\cot\theta\frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right) \quad \ldots (13)$$

and, the operator $\hat{L}^2$,

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$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \quad \ldots (14)$$

These are called Spherical Harmonics.

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Consider a particle with energy $E = p^2/2m$ moving along the x-axis. The uncertainty in its location is Δx. Show that $\Delta x \cdot \Delta p > \hbar$, then $\Delta E \cdot \Delta t > \hbar$ where $\left(\frac{p}{m}\right) \Delta t = \Delta x$.

$$E = p^2/2m \qquad \cancel{2} p \Delta p = \cancel{2} m \Delta E$$

$$p^2 = 2mE$$

$$p \Delta p = m \Delta E$$

$$\frac{p}{m} \Delta p = \Delta E$$

$$\Delta t \times \frac{p}{m} \Delta p = \Delta E \times \Delta t$$

$$\Delta x \times \Delta p = \Delta E \times \Delta t$$

Consider

$$E = \frac{p^2}{2m}$$

$$\Delta E = \frac{\cancel{2} p}{\cancel{2} m} \Delta p$$

$$\Delta E = \frac{p \Delta p}{m}$$

multiplying by Δt on both sides.

$$\Delta E \cdot \Delta t = \frac{p \Delta p \cdot \Delta t}{m} \qquad \left\{ \Delta x = \frac{p}{m} \Delta t \right\}$$

$$\Delta E \cdot \Delta t = \Delta x \cdot \Delta p \qquad \{ \Delta x \cdot \Delta p > \hbar \}$$

$$\Delta E \cdot \Delta t > \hbar$$

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Q1. Show that :-

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$$ in spherical polar coordinates.

Solution :- $\vec{L} = \vec{r} \times \vec{p}$ . . . (A)

$$L_x = yp_z - zp_y \quad \ldots \quad (1)$$

$$L_x = yp_z - zp_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y = zp_x - xp_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_z = xp_y - yp_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$L^2 \left(\equiv L_x^2 + L_y^2 + L_z^2\right)$$

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right\} \quad \ldots \quad (3)$$

Thus, $r^2 = x^2 + y^2 + z^2$ . . . (4)

$$\tan^2\theta = \frac{(x^2+y^2)}{z^2} \quad \ldots \quad (5)$$

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$$\tan\phi = \frac{y}{x} \quad \ldots (6)$$

Now,

$$2r\frac{\partial r}{\partial x} = 2x \quad \ldots (7)$$

$$2\tan\theta\sec^2\theta\frac{\partial \theta}{\partial x} = \frac{2x}{z^2} = \frac{2}{r}\tan\theta\sec\theta\cos\phi \quad \ldots (8)$$

$$\sec^2\phi\frac{\partial \phi}{\partial x} = \frac{-y}{x^2} = -\frac{1}{r}\operatorname{cosec}\theta\sec\phi\tan\phi \quad \ldots (9)$$

giving, $\frac{\partial r}{\partial x} = \sin\theta\cos\phi \quad \ldots (10)$

$$\frac{\partial \theta}{\partial x} = \frac{1}{r}\cos\theta\cos\phi \quad \ldots (11)$$

$$\frac{\partial \phi}{\partial x} = -\frac{1}{r}\sin\phi\operatorname{cosec}\theta \quad \ldots (12)$$

Similarly, we can obtain, $\frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial y}$ etc.

$$L_z\psi(r,\theta,\phi) = -i\hbar\left[x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x}\right]$$

$$= -i\hbar\left[x\left(\frac{\partial \psi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial \psi}{\partial \phi}\frac{\partial \phi}{\partial y}\right) - y\right.$$

$$\left.\left(\frac{\partial \psi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial \psi}{\partial \phi}\frac{\partial \phi}{\partial x}\right)\right]$$

$$= -i\hbar\Big[r\sin\theta\cos\phi\Big\{\frac{\partial\psi}{\partial r}(\sin\theta\sin\phi) + \frac{\partial\psi}{\partial\theta}\Big(\frac{1}{r}\cos\theta\sin\phi\Big) + \frac{\partial\psi}{\partial\phi}\Big(\frac{1}{r}\cos\phi\,\mathrm{cosec}\,\theta\Big)\Big\} - r\sin\theta\sin\phi\Big\{\frac{\partial\psi}{\partial r}\times(\sin\theta\cos\phi) + \frac{\partial\psi}{\partial\theta}\Big(\frac{1}{r}\cos\theta\cos\phi\Big) + \frac{\partial\psi}{\partial\phi}\Big(-\frac{1}{r}\sin\phi\,\mathrm{cosec}\,\theta\Big)\Big\}\Big] \quad \ldots (13)$$

$$\therefore L_z\psi(r,\theta,\phi) = -i\hbar\frac{\partial\psi(r,\theta,\phi)}{\partial\phi} \quad \ldots (14)$$

$$\therefore L_z = -i\hbar\frac{\partial}{\partial\phi} \quad \ldots (15)$$

Q2. Show that :-

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

in spherical polar coordinates.

Q3. Show that :-

(1) $[e^{p}, \hat{p}] = 0$

(2) $[x, \hat{p}^2] = 2i\hbar\hat{p}$

A- (1) $[e^{p}, \hat{p}] = \left[\sum_{n=0}^{\infty}\frac{\hat{p}^n}{n!}, \hat{p}\right]$

$$= \sum\frac{1}{n!}[\hat{p}^n, \hat{p}]$$

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$$= [1, \hat{p}] + [\hat{p}, \hat{p}] + \frac{1}{2!}[\hat{p}^2, \hat{p}] + \cdots = 0$$

i.e,

$$[e^{\hat{p}}, \hat{p}]\, g(x) = \left[\exp\left(-\frac{i\hbar\,\partial}{\partial x}\right) - \frac{i\hbar\,\partial}{\partial x}\right] g(x) = 0$$

A-(2) $[\hat{x}, \hat{p}]\, g(x) = i\hbar\left(-x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}x\right) g(x)$

$$= i\hbar\left(-x\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial x} + g\right) = i\hbar\, g(x)$$

$$\therefore \boxed{[\hat{x}, \hat{p}] = i\hbar}$$

$$[\hat{x}, \hat{p}^2] = [\hat{x}, \hat{p}]\hat{p} + \hat{p}[\hat{x}, \hat{p}]$$

$$\boxed{[\hat{x}, \hat{p}^2] = 2i\hbar\hat{p}}$$

$$[\hat{x}, \hat{p}^2]\, g(x) = 2\hbar^2 \frac{\partial g}{\partial x}$$

B

$$[\hat{x}^2, \hat{p}] = \hat{x}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{x}$$

$$= 2i\hbar\hat{x}$$

$$\boxed{[\hat{x}^2, \hat{p}] = 2i\hbar x}$$

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